

GENERALIZED G_2 -MANIFOLDS AND $SU(3)$ -STRUCTURES

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ABSTRACT. We construct a family of compact 7-dimensional manifolds endowed with a weakly integrable generalized G_2 -structure with respect to a closed and non-zero 3-form. We relate the previous structures with $SU(3)$ -structures in dimension 7. Moreover, we investigate which types of $SU(3)$ -structures on a 6-dimensional manifold N give rise to a strongly integrable generalized G_2 -structure with respect to a non-zero 3-form on the product $N \times S^1$.

1. INTRODUCTION

The notion of generalized geometry goes back to the work of Hitchin [12] (see also [13]). In this context, Witt [17] introduced a new type of structures on a 7-dimensional manifold M in terms of a differential form of mixed degree, thus generalizing the classical notion of G_2 -structure determined by a stable and positive 3-form. Instead of studying geometry on the tangent bundle TM of the manifold, one considers the bundle $TM \oplus T^*M$ endowed with a natural orientation and an inner product of signature $(7, 7)$, where T^*M denotes the cotangent bundle of M . In this way, if M is spin, then the differential form of mixed type can be viewed as a $G_2 \times G_2$ -invariant spinor ρ for the bundle and it is called the structure form.

These structures are called *generalized G_2 -structures* and they induce a Riemannian metric, a 2-form b (the B -field), two unit spinors Ψ_{\pm} and a function ϕ (the dilaton). By [17], any $G_2 \times G_2$ -invariant spinor ρ is stable and has a canonical expression by $\rho = e^{-\phi} e^{\frac{1}{2}} \wedge (\Psi_+ \otimes \Psi_-)^{ev, od}$ in terms of the two spinors, the B -field and the dilaton function. In the paper we will restrict to the case of constant dilaton, i.e. $\phi = \text{const}$, and trivial B -field.

Up to a B -field transformation, a generalized G_2 -structure is essentially a pair of G_2 -structures. If the two spinors Ψ_+ and Ψ_- are linearly independent, then the intersection of the two isotropy groups, both isomorphic to G_2 , determined by the two spinors coincides with $SU(3)$. Therefore, one can express the structure form in terms of the form α dual to the unit vector stabilized by $SU(3)$ and of the forms $(\omega, \psi = \psi_+ + i\psi_-)$, associated with $SU(3)$, where ω is the fundamental form and ψ is the complex volume form. Assuming that the angle between Ψ_+ and Ψ_- is $\frac{\pi}{2}$, then it turns out that

$$(1) \quad \begin{aligned} \rho &= (\Psi_+ \otimes \Psi_-)^{ev} = \omega + \psi_+ \wedge \alpha - \frac{1}{6} \omega^3 \wedge \alpha, \\ \hat{\rho} &= (\Psi_+ \otimes \Psi_-)^{od} = \alpha - \psi_- - \frac{1}{2} \omega^2 \wedge \alpha, \end{aligned}$$

Date: February 2, 2008.

1991 *Mathematics Subject Classification.* 53C10, 53C25, 83E50.

This work was supported by the Projects MIUR “Geometric Properties of Real and Complex Manifolds”, “Riemannian Metrics and Differentiable Manifolds” and by GNSAGA of INdAM.

where $\hat{\rho}$ is the companion of ρ and ω^k denotes the k -power wedge of ω . In this paper we will consider generalized G_2 -structures defined by the previous structure forms. In this case, the two associated G_2 -structures do not coincide.

If H is a 3-form (not necessarily closed) on M , then one can consider two types of generalized G_2 -structures with respect to the 3-form H : the *strongly integrable* ones, i.e. those associated to a structure form ρ which satisfies

$$(2) \quad d\rho + H \wedge \rho = d\hat{\rho} + H \wedge \hat{\rho} = 0,$$

and the *weakly integrable* ones, i.e. those defined by the condition

$$d\rho + H \wedge \rho = \lambda \hat{\rho},$$

where λ is a non-zero constant. The previous structures are said of even or odd type according to the parity of ρ .

Note that these definitions of integrability are slightly different from the ones given in [17], where the closure of the 3-form H is assumed.

If H is closed, then the twisted operator $d_H \cdot = d \cdot + H \wedge \cdot$ defines a differential complex and if, in addition, M is compact, then the strongly integrable generalized G_2 -structures can be interpreted as critical points of a certain functional [17, Theorem 4.1]. In this case the underlying spinors Ψ_{\pm} are parallel with respect to the Levi-Civita connection and therefore there exist no non-trivial compact examples with such structures, i.e. there are only the classical examples of manifolds with holonomy contained in G_2 . If H is not closed, then we will show that compact examples can be constructed starting from a 6-dimensional manifold endowed with an $SU(3)$ -structure.

If H is closed, then the weakly integrable generalized G_2 -structures can be also viewed as critical points of a functional under a constraint, but they have no classical counterpart. The existence of weakly integrable generalized G_2 -structures with respect to a closed 3-form H on a compact manifold was posed as an open problem in [17]. We construct such structures on a family of compact manifolds and we relate them with $SU(3)$ -structures in dimension 7, where $SU(3)$ is identified with the subgroup $SU(3) \times \{1\}$ of $SO(7)$.

After reviewing the general theory of generalized G_2 -structures, in section 3 we construct a family of compact 7-dimensional manifolds endowed with a weakly integrable generalized G_2 -structure with respect to a closed and non-zero 3-form H (Theorem 3.1). The corresponding structure form is the odd type form $\hat{\rho}$ given by (1). These manifolds are obtained as a compact quotients M_{β} by uniform discrete subgroups (parametrized by the p -th roots of unity $e^{i\beta}$) of a semi-direct product $SU(2) \ltimes \mathbb{H}$, where \mathbb{H} denotes the quaternions. It turns out that these manifolds have an $SU(3)$ -structure (ω, η, ψ) such that

$$(3) \quad d\eta = \lambda\omega, \quad d(\eta \wedge \psi_{\pm}) = 0.$$

In particular they are contact metric. The structures satisfying the condition (3) can arise on hypersurfaces of 8-dimensional manifolds with an integrable $SU(4)$ -structure and they are the analogous of the “hypo” $SU(2)$ -structures in dimension 5 (see [6]). In the same vein of [12], we consider a family $(\omega(t), \eta(t), \psi(t))$ of $SU(3)$ -structures containing the $SU(3)$ -structure (ω, η, ψ) and the corresponding evolution equations. In this way in section 4 we show that on the product of M_{β} with an open interval there exists a Riemannian metric with discrete holonomy contained in $SU(4)$ (Theorem 4.1).

Starting from a 6-dimensional manifold N endowed with an $SU(3)$ -structure (ω, g, ψ) , it is possible to define in a natural way a generalized G_2 -structure with the structure form ρ of even type given by (1) on the Riemannian product $(M = N \times S^1, h)$, with

$$h = g + dt \otimes dt$$

and $\alpha = dt$. In [17] an example of this type with a 6-dimensional nilmanifold N was considered in order to construct a compact manifold endowed with a strongly integrable generalized G_2 -structure with respect to a non-closed 3-form H .

We will prove in general that if N is a 6-dimensional manifold endowed with an $SU(3)$ -structure (ω, g, ψ) , then the generalized G_2 -structure defined by ρ on $N \times S^1$ satisfies the conditions (2), for a non-zero 3-form H , if and only if

$$(4) \quad d\omega = 0, \quad d\psi_+ = -\pi_2 \wedge \omega, \quad d\psi_- = 0,$$

where the 2-form π_2 is the unique non zero component of the intrinsic torsion (see Theorem 5.1). We will call $SU(3)$ -structures which satisfy the previous conditions belonging to the class \mathcal{W}_2^+ . The 3-form H is related to the component π_2 of the intrinsic torsion by $H = \pi_2 \wedge \alpha$ and we will show that H will never be closed unless $\pi_2 = 0$.

It has to be noted that, if (ω, g, ψ) is in the class \mathcal{W}_2^+ , then the $SU(3)$ -structure given by $(\omega, g, i\psi)$ is symplectic half-flat (see [5]), i.e. the fundamental form ω and the real part of the complex volume form are both closed. The half-flat structures turn out to be useful in the construction of metrics with holonomy group contained in G_2 (see e.g. [12, 5, 4]). Indeed, starting with a half-flat structure on N , if certain evolution equations are satisfied, then there exists a Riemannian metric with holonomy contained in G_2 on the product of the manifold N with some open interval. Examples of compact manifolds with symplectic half-flat structures have been given in [7], where invariant symplectic half-flat structures on nilmanifolds are classified. Other examples are considered in [8] where Lagrangian submanifolds are studied instead.

Acknowledgements The authors thank Simon Salamon, Frederik Witt for useful comments and suggestions. They also thank the Centro di Ricerca Matematica “Ennio De Giorgi”, Pisa, for the warm hospitality.

2. GENERALIZED G_2 STRUCTURES AND SPINORS

In this section we are going to recall some facts on generalized G_2 -structures which have been studied by Jeschek and Witt in [17, 18, 14] in the general case of ϕ non-constant and non-trivial B -field. In the next sections we will deal with the case $\phi = \text{const}$ and trivial B -field.

Let V be a 7-dimensional real vector space and denote by V^* the dual space of V . Then $V \oplus V^*$ has a natural orientation and a inner product of signature $(7, 7)$ defined by

$$(v + \xi, v + \xi) = -\frac{1}{2}\xi(v), \quad \forall v \in V, \xi \in V^*.$$

The inner product determines a group conjugate to $SO(7, 7)$ inside the linear group $GL(14)$. Since as $GL(7)$ -space $\mathfrak{so}(7, 7) = \text{End}(V) \oplus \Lambda^2 V^* \oplus \Lambda^2 V$, any $b \in \Lambda^2 V^*$ defines an element (called *B-field*) in $\mathfrak{so}(7, 7)$. By exponentiating to $SO(7, 7)$ the action of $\Lambda^2 V^* \subset \mathfrak{so}(7, 7)$

$$v \rightarrow v \lrcorner b,$$

one gets an action on $V \oplus V^*$, given by $\exp(b)(v \oplus \xi) = v \oplus (v \lrcorner b + \xi)$. Then $V \oplus V^*$ acts on $\Lambda^* V^*$ by

$$(v + \xi)\eta = \iota(v)\eta + \xi \wedge \eta,$$

and we have

$$(v + \xi)^2 \eta = -(v + \xi, v + \xi)\eta.$$

Therefore $\Lambda^* V^*$ can be viewed as a module over the Clifford algebra of $V \oplus V^*$. The space $\Lambda^* V^*$, as the spin representation of $Spin(7, 7)$, determines the splitting of $\Lambda^* V^* \otimes (\Lambda^7 V)^{\frac{1}{2}}$

$$\begin{aligned} S^+ &= \Lambda^{ev} V^* \otimes (\Lambda^7 V)^{\frac{1}{2}} \\ S^- &= \Lambda^{od} V^* \otimes (\Lambda^7 V)^{\frac{1}{2}} \end{aligned}$$

into the sum of the two irreducible spin representations. By considering $b \in \Lambda^2 V^*$, then one has the following induced action on spinors given by

$$\exp(b)\eta = (1 + b + \frac{1}{2}b \wedge b + \dots) \wedge \eta = e^b \wedge \eta.$$

If σ is the Clifford algebra anti-automorphism defined by $\sigma(\gamma^p) = \epsilon(p)\gamma^p$, on any element of degree p , with

$$\epsilon(p) = \begin{cases} 1 & \text{for } p \equiv 0, 3 \pmod{4}, \\ -1 & \text{for } p \equiv 1, 2 \pmod{4}, \end{cases}$$

then S^+ and S^- are totally isotropic with respect to the symmetric bilinear form $q(\alpha, \beta)$ defined as the top degree component of $\alpha \wedge \sigma(\beta)$ (see [17]).

A *generalized G_2 -structure* on a 7-dimensional manifold M is a reduction from the structure group $\mathbb{R}^* \times Spin(7, 7)$ of the bundle $TM \oplus T^*M$ to $G_2 \times G_2$. Such a structure determines a generalized oriented metric structure (g, b) , (i.e. a Riemannian metric g , a B -field b and an orientation on V) and a real scalar function ϕ (the *dilaton*). Therefore we get a pair of two G_2 -structures associated with two unit spinors Ψ_{\pm} in the irreducible spin representation $\Delta = \mathbb{R}^8$ of $Spin(7)$. There is, up to a scalar, a unique invariant in $\Lambda^{ev} V^* \otimes \Lambda^{od} V^*$, given by the box operator

$$\square_{\rho} : \Lambda^{ev, od} V^* \rightarrow \Lambda^{od, ev} V^*, \quad \tilde{\rho} \rightarrow e^{\frac{b}{2}} \wedge *_g \sigma(e^{-\frac{b}{2}} \wedge \tilde{\rho}).$$

If ρ is a $G_2 \times G_2$ -invariant spinor, then its *companion* $\hat{\rho} = \square_{\rho} \rho$ is still a $G_2 \times G_2$ -invariant spinor. To any $G_2 \times G_2$ -invariant spinor ρ one can associate a volume form \mathcal{Q} defined by

$$(5) \quad \mathcal{Q} : \rho \rightarrow q(\hat{\rho}, \rho).$$

Using the isomorphism $\Delta \otimes \Delta \cong \Lambda^{ev, od}$, Witt in [17, Proposition 2.4] derived the following normal form for $[\Psi_+ \otimes \Psi_-]^{ev, od}$ in terms of a suitable orthonormal basis (e^1, \dots, e^7) , namely

$$\begin{aligned} (\Psi_+ \otimes \Psi_-)^{ev} &= \cos(\theta) + \sin(\theta)(e^{12} + e^{34} + e^{56}) + \\ &\quad \cos(\theta)(-e^{1367} - e^{1457} - e^{2357} + e^{2467} - e^{1234} - e^{1256} - e^{3456}) + \\ &\quad \sin(\theta)(e^{1357} - e^{1467} - e^{2367} - e^{2457}) - \sin(\theta)e^{123456}, \\ (\Psi_+ \otimes \Psi_-)^{od} &= \sin(\theta)e^7 + \sin(\theta)(-e^{136} - e^{145} - e^{235} + e^{246}) + \\ &\quad \cos(\theta)(-e^{127} - e^{347} - e^{567} - e^{135} + e^{146} + e^{236} + e^{245}) + \\ &\quad \sin(\theta)(-e^{12347} - e^{12567} - e^{34567}) + \cos(\theta)e^{1234567}, \end{aligned}$$

where θ is the angle between Ψ_+ and Ψ_- and $e^{i \dots j}$ denotes the wedge product $e^i \wedge \dots \wedge e^j$.

If the spinors Ψ_+ and Ψ_- are linearly independent, then (see Corollary 2.5 of [17])

$$\begin{aligned} (\Psi_+ \otimes \Psi_-)^{ev} &= \cos(\theta) + \sin(\theta)\omega - \cos(\theta)(\psi_- \wedge \alpha + \frac{1}{2}\omega^2) \\ &\quad + \sin(\theta)\psi_+ \wedge \alpha - \frac{1}{6}\sin(\theta)\omega^3, \\ (\Psi_+ \otimes \Psi_-)^{od} &= \sin(\theta)\alpha - \cos(\theta)(\psi_+ + \omega \wedge \alpha) - \sin(\theta)\psi_- \\ &\quad - \frac{1}{2}\sin(\theta)\omega^2 \wedge \alpha + \cos(\theta)\text{vol}_g, \end{aligned}$$

where α denotes the dual of the unit vector in V , stabilized by $SU(3)$,

$$\omega = e^{12} + e^{34} + e^{56}$$

is the fundamental form and ψ_{\pm} are the real and imaginary parts respectively of the complex volume form

$$\psi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6).$$

A $G_2 \times G_2$ -invariant spinor ρ is stable in the sense of Hitchin (see [13]), i.e. ρ lies in an open orbit under the action of $\mathbb{R}^+ \times Spin(7, 7)$.

By [17, Theorem 2.9] the generalized G_2 -structures are in 1 – 1 correspondence with lines of spinors ρ in Λ^{ev} (or Λ^{od}) whose stabilizer under the action of $Spin(7, 7)$ is isomorphic to $G_2 \times G_2$.

The spinor ρ is called the *structure form* of the generalized G_2 structure and it can be uniquely written (modulo a simultaneous change of sign for Ψ_+ and Ψ_-) as

$$\rho = e^{-\phi}(\Psi_+ \otimes \Psi_-)_b^{ev},$$

where b is the B -field, $\Psi_{\pm} \in \Delta$ are two unit spinors, the function ϕ is the dilaton and the subscript b denotes the wedge with the exponential $e^{\frac{\phi}{2}}$.

A (*topological*) *generalized G_2 -structure* over M is a topological $G_2 \times G_2$ -reduction of the $SO(7, 7)$ -principal bundle associated with $TM \oplus T^*M$ and it is characterized by a stable even or odd spinor ρ which can be viewed as a form. This is equivalent to say that there exists an $SO(7)$ -principal fibre bundle which has two G_2 -subbundles (or equivalently two G_2^{\pm} -structures).

In the sequel we will omit topological when we will refer to a generalized G_2 -structure.

Let H be a 3-form and λ be a real, non-zero constant. A generalized G_2 -structure (M, ρ) is called *strongly integrable* with respect to H if

$$d_H \rho = 0, \quad d_H \hat{\rho} = 0,$$

where $d_H \cdot = d \cdot + H \wedge \cdot$ is the twisted operator of d . By [17] there are no non-trivial compact examples with a strongly integrable generalized G_2 -structure with respect to a closed 3-form H .

If

$$d_H \rho = \lambda \hat{\rho},$$

then the generalized G_2 -structure is said to be *weakly integrable* of *even* or *odd* type according to the parity of the form ρ . The constant λ (called the *Killing number*) is the 0-torsion form of the two underlying G_2 -structures. Indeed, by Corollary 4.6 of [17], there exist two unique determined linear connections ∇^{\pm} , preserving the two G_2^{\pm} -structures, with skew-symmetric torsion $\pm T = \frac{1}{2}db + H$. If the structure

is of odd type, then

$$\begin{aligned} d\varphi_+ &= \frac{12}{7}\lambda * \varphi_+ + \frac{3}{2}d\phi \wedge \varphi_+ - *T_{27}^+, \\ d * \varphi_+ &= 2d\phi \wedge * \varphi_+ \end{aligned}$$

and

$$\begin{aligned} d\varphi_- &= \frac{12}{7}\lambda * \varphi_- + \frac{3}{2}d\phi \wedge \varphi_- - *T_{27}^-, \\ d * \varphi_- &= 2d\phi \wedge * \varphi_-, \end{aligned}$$

where T_{27}^\pm denotes the component of T into the 27-dimensional irreducible G_2^\pm -module

$$\Lambda_{27}^{3\pm} = \{\gamma \in \Lambda^3 \mid \gamma \wedge \varphi_+ = \gamma \wedge \varphi_- = 0\}.$$

This is equivalent to say that $e^{-\phi}[\Psi_+ \otimes \Psi_-]$ satisfies the generalized Killing and dilatino equation (see [17, 10]).

In both cases there is a characterization in terms of the two metric connections ∇^\pm with skew symmetric torsion $\pm T$ (see [17, Theorem 4.3]). Indeed, a generalized G_2 -manifold (M, ρ) is weakly integrable with respect to H if and only if

$$\nabla^{LC} \Psi_\pm \pm \frac{1}{4}(X \lrcorner T) \cdot \Psi_\pm = 0,$$

where ∇^{LC} is the Levi-Civita connection, $X \lrcorner$ denotes the contraction by X and the following additional conditions are satisfied

$$\left(d\phi \pm \frac{1}{2}(X \lrcorner T) \pm \lambda \right) \cdot \Psi_\pm = 0,$$

if ρ is of even type or

$$\left(d\phi \pm \frac{1}{2}(X \lrcorner T) + \lambda \right) \cdot \Psi_\pm = 0,$$

if ρ is of odd type. Taking $\lambda = 0$ above equations yield strong integrability with respect to H , instead.

Examples of generalized G_2 -structures are given by the *straight* generalized G_2 -structures, i.e. structures defined by one spinor $\Psi = \Psi_+ = \Psi_-$. These structures are induced by a classical G_2 -structure (M, φ) and are strongly integrable with respect to a closed 3-form T only if the holonomy of the metric associated with φ is contained in G_2 .

If H is closed, then it has to be noted that, in the compact case, the structure form ρ of a strongly integrable generalized G_2 -structure corresponds to a critical point of a functional on stable forms. Indeed, since stability is an open condition, if M is compact then one can consider the functional

$$V(\rho) = \int_M \mathcal{Q}(\rho),$$

where \mathcal{Q} is defined as in (5). By [17, Theorem 4.1] a d_H -closed stable form ρ is a critical point in its cohomology class if and only if $d_H \hat{\rho} = 0$.

Again in the compact case a d_H -exact form $\hat{\rho} \in \Lambda^{ev, od}(M)$ is a critical point of the functional V under some constraint if and only if $d_H \rho = \lambda \hat{\rho}$, for a real non zero constant λ .

3. COMPACT EXAMPLES OF WEAKLY INTEGRABLE MANIFOLDS

In this section we will construct examples of compact manifolds endowed with a weakly integrable generalized G_2 -structure with respect to a closed 3-form H .

Consider the 7-dimensional Lie algebra \mathfrak{g} with structure equations:

$$\begin{cases} de^1 = ae^{46}, \\ de^2 = -\frac{1}{2}ae^{36} - \frac{1}{2}ae^{45} + \frac{1}{2}ae^{17}, \\ de^3 = -\frac{1}{2}ae^{15} + \frac{1}{2}ae^{26} - \frac{1}{2}ae^{47}, \\ de^4 = -ae^{16}, \\ de^5 = \frac{1}{2}ae^{13} - \frac{1}{2}ae^{24} - \frac{1}{2}ae^{67}, \\ de^6 = ae^{14}, \\ de^7 = -\frac{1}{2}ae^{12} - \frac{1}{2}ae^{34} - \frac{1}{2}ae^{56}, \end{cases}$$

where a is a real parameter different from zero.

It can be easily checked that the Lie algebra \mathfrak{g} is not solvable since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and that it is unimodular. We can also view \mathfrak{g} as the semidirect sum

$$\mathfrak{g} = \mathfrak{su}(2) \oplus_{\delta} \mathbb{R}^4,$$

where

$$\mathfrak{su}(2) = \text{span} \langle e_1, e_4, e_6 \rangle, \quad \mathbb{R}^4 = \text{span} \langle e_2, e_3, e_5, e_7 \rangle$$

and $\delta : \mathfrak{su}(2) \rightarrow \mathfrak{Der}(\mathbb{R}^4)$ is given by

$$\begin{aligned} \delta(e_1) = ad_{e_1} &= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}a \\ 0 & 0 & \frac{1}{2}a & 0 \\ 0 & -\frac{1}{2}a & 0 & 0 \\ \frac{1}{2}a & 0 & 0 & 0 \end{pmatrix}, \\ \delta(e_4) = ad_{e_4} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}a & 0 \\ 0 & 0 & 0 & \frac{1}{2}a \\ -\frac{1}{2}a & 0 & 0 & 0 \\ 0 & -\frac{1}{2}a & 0 & 0 \end{pmatrix}, \\ \delta(e_6) = ad_{e_6} &= \begin{pmatrix} 0 & -\frac{1}{2}a & 0 & 0 \\ \frac{1}{2}a & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}a \\ 0 & 0 & -\frac{1}{2}a & 0 \end{pmatrix}. \end{aligned}$$

If we identify \mathbb{R}^4 with the space \mathbb{H} of quaternions, then

$$ad_{e_1} = \frac{1}{2}aL_k, \quad ad_{e_4} = \frac{1}{2}aL_{-j}, \quad ad_{e_6} = \frac{1}{2}aL_i,$$

where L_q denotes the left multiplication by the quaternion q .

Therefore, the product on the corresponding Lie group $G = SU(2) \ltimes \mathbb{H}$, for $a = 2$, is given by

$$(A, q) \cdot (A', q') = (AA', Aq' + q), \quad A, A' \in SU(2), \quad q, q' \in \mathbb{H},$$

where we identify $SU(2)$ with the group of quaternions of unit norm.

Theorem 3.1. *The Lie group $G = SU(2) \ltimes \mathbb{H}$ admits compact quotients $M_{\beta} = G/\Gamma_{\beta}$, with $e^{i\beta}$ primitive p -th root of unity (p prime), and M_{β} has an invariant weakly integrable generalized G_2 -structure with respect to a closed 3-form H .*

Proof. Consider the discrete subgroup $\Gamma_\beta = \langle A_\beta \rangle \ltimes \mathbb{Z}^4$, where $\langle A_\beta \rangle$ is the subgroup of $SU(2)$ generated by

$$A_\beta = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix},$$

with $e^{i\beta}$ primitive p -th root of unity and p prime.

Then one can check that Γ_β is a closed subgroup of G . Let (A', q') be any point of G . Thus

$$[(A', q')] = \{(A_\beta^m A', A_\beta^m q' + r), m \in \mathbb{Z}, r \in \mathbb{Z}^4\}$$

is the equivalence class of (A', q') . In particular, $[(A', q')] = [(A', q' + r)]$ and therefore the restriction of the projection $\pi : G \rightarrow G/\Gamma_\beta$ to $SU(2) \times [0, 1]^4$ is surjective.

Then the quotient $M_\beta = (SU(2) \ltimes \mathbb{H})/\Gamma_\beta$ is a compact manifold.

Consider the invariant metric g on M_β such that the basis (e^1, \dots, e^7) is orthonormal and take the generalized G_2 structure defined by the structure form of odd type

$$\rho = e^7 - e^{136} - e^{145} - e^{235} + e^{246} - e^{12347} - e^{12567} - e^{34567},$$

in terms of the basis (e^1, \dots, e^7) . The companion of ρ is

$$\hat{\rho} = e^{12} + e^{34} + e^{56} + e^{1357} - e^{1467} - e^{2367} - e^{2457} - e^{123456}.$$

Then the structure form ρ defines a weakly integrable generalized G_2 -structure with respect to a closed 3-form H , i.e. $d_H \rho = \lambda \hat{\rho}$ (λ non-zero constant), if and only if

$$(6) \quad \begin{cases} de^7 = \lambda \omega, \\ d\psi_- = (H - \lambda \psi_+) \wedge e^7, \\ H \wedge \psi_- = -\frac{1}{3} \lambda \omega^3, \end{cases}$$

where ω, ψ_\pm are given by

$$(7) \quad \begin{cases} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}. \end{cases}$$

The equations (6) are satisfied with $\lambda = -\frac{1}{2}a$ and

$$H = -ae^{146}.$$

□

Observe that H is also co-closed, i.e. $d * H = 0$. Moreover, if $a \leq 1$, H is a calibration in the sense of [11].

In this way we get compact examples with a weakly integrable generalized G_2 -structure with respect to the closed 3-form H . The induced invariant metric on M_β is not flat, since the inner product

$$g = \sum_{i=1}^7 (e^i)^2$$

on the Lie algebra \mathfrak{g} is not flat. Indeed, the Ricci tensor of g is diagonal with respect to the orthonormal basis (e_1, \dots, e_7) and its non zero components are given by:

$$Ric(e_1, e_1) = \frac{1}{2}a^2 = Ric(e_4, e_4) = Ric(e_6, e_6).$$

4. LINK WITH $SU(3)$ -STRUCTURES IN DIMENSION 7 AND EVOLUTION EQUATIONS

In this section we will relate the weakly integrable generalized G_2 -structures constructed in the previous section with $SU(3)$ -structures in dimension 7.

Since the 1-form $\eta = e^7$ is a contact form on the Lie algebra \mathfrak{g} , then M_β is a contact metric manifold. Moreover, by (6) M_β has an $SU(3)$ -structure defined by $(\omega, \eta, \psi = \psi_+ + i\psi_-)$ such that

$$(8) \quad \begin{cases} d\omega = 0, \\ d(\psi_\pm \wedge \eta) = 0. \end{cases}$$

Here we identify $SU(3)$ as the subgroup $SU(3) \times \{1\}$ of $SO(7)$.

Note that the $SU(3)$ -structures $(\omega, \eta, \psi = \psi_+ + i\psi_-)$ on 7-dimensional manifolds for which $d\omega = 0$ and $d(\psi_\pm) = 0$ where considered in [16]. In this case one cannot find any closed 3-form H such that conditions (8) are satisfied since H has to be equal to $\lambda\psi_+$ and the third equation cannot hold. It would be interesting to investigate if there are other 7-dimensional examples endowed with an $SU(3)$ -structures which satisfy the conditions (8) and giving rise to a weakly integrable G_2 -structure with respect to a closed 3-form H .

In general, let $\iota : M^7 \rightarrow N^8$ be an embedding of a an oriented 7-manifold M^7 into a 8-manifold N^8 with unit normal vector V . Then an $SU(4)$ -structure $(\tilde{\omega}, \tilde{g}, \tilde{\psi})$ (or equivalently a special almost Hermitian structure, see e.g. [3]), where $(\tilde{\omega}, \tilde{g})$ is a $U(4)$ -structure and $\tilde{\psi} = \tilde{\psi}_+ + i\tilde{\psi}_-$ is complex 4-form of unit norm, defines in a natural way an $SU(3)$ -structure $(\omega, \eta, g, \psi = \psi_+ + i\psi_-)$ on M^7 given by:

$$\eta = -V \lrcorner \tilde{\omega}, \quad \omega = \iota^* \tilde{\omega}, \quad g = \iota^* \tilde{g}, \quad \psi_+ = -V \lrcorner \tilde{\psi}_+, \quad \psi_- = V \lrcorner \tilde{\psi}_-.$$

Then, if γ denotes the 1-form dual to V , then we have

$$\begin{aligned} \tilde{\omega} &= \omega + \eta \wedge \gamma, \\ \tilde{\psi} &= (\psi_+ + i\psi_-) \wedge (\eta + i\gamma). \end{aligned}$$

The integrability of the $SU(4)$ -structure $(\tilde{\omega}, \tilde{g}, \tilde{\psi})$ implies conditions (8), which can be viewed as the analogous of the equations defining the hypo $SU(2)$ -structures in dimension 5 (see [5]).

Vice versa, given an $SU(3)$ -structure (ω, η, ψ) on M^7 , an $SU(4)$ -structure on $M^7 \times \mathbb{R}$ is defined by

$$(9) \quad \begin{aligned} \tilde{\omega} &= \omega + \eta \wedge dt, \\ \tilde{\psi} &= \psi \wedge (\eta + idt), \end{aligned}$$

where t is a coordinate on \mathbb{R} .

If the $SU(3)$ -structure (ω, η, ψ) on M^7 belongs to a one-parameter family of $SU(3)$ -structures $(\omega(t), \eta(t), \psi(t))$ satisfying the equations (8) and such that

$$(10) \quad \begin{cases} \partial_t \omega(t) = -\hat{d}\eta(t), \\ \partial_t(\psi_+(t) \wedge \eta(t)) = \hat{d}\psi_-(t), \\ \partial_t(\psi_-(t) \wedge \eta(t)) = -\hat{d}\psi_+(t), \end{cases}$$

for all $t \in (b, c)$, where ∂_t denotes the derivative with respect to t and \hat{d} is the exterior differential on M^7 , then the $SU(4)$ -structure given by (9) on $M^7 \times (b, c)$ is integrable, i.e. $\tilde{\omega}$ and $\tilde{\psi}$ are both closed. In particular, the associated Riemannian metric on $M^7 \times (b, c)$ has holonomy contained in $SU(4)$ and consequently it is Ricci-flat.

For the manifolds M_β a solution of the evolution equations (10) is given by

$$\begin{aligned} \omega(t) &= u(t)v(t)(e^{12} + e^{34} + e^{56}), \\ \psi_+(t) &= u(t)v(t)^2(e^{135} - e^{236} - e^{245}) - u(t)^3e^{146}, \\ \psi_-(t) &= u(t)^2v(t)(e^{136} + e^{145} - e^{246}) + v(t)^3e^{235}, \\ \eta(t) &= \frac{1}{v(t)^3}e^7, \end{aligned}$$

where $u(t), v(t)$ solve the system of ordinary differential equations

$$\begin{cases} \frac{d}{dt}(u(t)v(t)) = \frac{1}{2}a\frac{1}{v(t)^3}, \\ \frac{d}{dt}\left(\frac{u(t)}{v(t)}\right) = \frac{1}{2}av(t)^3, \end{cases}$$

such that $u(0) = v(0) = 1$. The previous system is equivalent to

$$(11) \quad \begin{cases} u'(t) = \frac{1}{4}a\left(\frac{1}{v(t)^4} + v(t)^4\right), \\ v'(t) = \frac{1}{4}a\left(\frac{1}{u(t)v(t)^3} - \frac{v(t)^5}{u(t)}\right). \end{cases}$$

Then, by the theorem on existence of solutions for a system of ordinary differential equations, one can show that on a open interval (b, c) containing $t = 0$ the system (11) admits a unique solution $(u(t), v(t))$ satisfying the initial condition $u(0) = v(0) = 1$. Actually, the solution is given by

$$u(t) = 1 + \frac{1}{2}at, \quad v(t) = 1.$$

Hence, we can prove the following

Theorem 4.1. *On the product of M_β with some open interval (b, c) there exists a Riemannian metric with discrete holonomy contained in $SU(4)$.*

Proof. The basis of 1-forms on the manifold $M_\beta \times (b, c)$ given by

$$\begin{aligned} E^1 &= (1 + \frac{1}{2}at)e^1, \quad E^2 = e^2, \quad E^3 = (1 + \frac{1}{2}at)e^3, \quad E^4 = (1 + \frac{1}{2}at)e^4, \\ E^5 &= e^5, \quad E^6 = (1 + \frac{1}{2}at)e^6, \quad E^7 = e^7, \quad E^8 = dt \end{aligned}$$

is orthonormal with respect to the Riemannian metric with holonomy contained in $SU(4)$. By a direct computation we have that the non zero Levi-Civita connection 1-forms are given by

$$\begin{aligned}\theta_4^1 &= -\theta_3^2 = \theta_7^5 = \theta_8^6 = \frac{a}{2+at}E^6, \\ \theta_6^1 &= -\theta_5^2 = -\theta_7^3 = -\theta_8^4 = -\frac{a}{2+at}E^4, \\ \theta_8^1 &= -\theta_7^2 = \theta_5^3 = \theta_6^4 = \frac{a}{2+at}E^1.\end{aligned}$$

Therefore, all the curvature forms Ω_j^i vanish and consequently the holonomy algebra is trivial. \square

5. STRONG INTEGRABILITY AND $SU(3)$ -STRUCTURES IN DIMENSION 6

In this section we are going to consider the structure form ρ of even type

$$(12) \quad \rho = \omega + \psi_+ \wedge \alpha - \frac{1}{6}\omega^3$$

on the product of a 6-dimensional manifold N endowed with an $SU(3)$ -structure cross S^1 . We will investigate which type of $SU(3)$ -structures give rise to a strongly integrable generalized G_2 -structure with respect to a non-zero 3-form.

Let N be a 6-dimensional manifold. An $SU(3)$ -structure on N is determined by a Riemannian metric g , an orthogonal almost complex structure J and a choice of a complex volume form $\psi = \psi_+ + i\psi_-$ of unit norm. We will denote by (ω, ψ) an $SU(3)$ -structure, where ω is the fundamental form defined by

$$\omega(X, Y) = g(JX, Y),$$

for any pair of vector fields X, Y on N . Locally one may choose an orthonormal basis (e^1, \dots, e^6) of the vector cotangent space T^* such that ω and ψ_{\pm} are given by (7).

These forms satisfy the following compatibility relations

$$\omega \wedge \psi_{\pm} = 0, \quad \psi_+ \wedge \psi_- = \frac{2}{3}\omega^3.$$

The intrinsic torsion of the $SU(3)$ -structure belongs to the space (see [5])

$$T^* \otimes \mathfrak{su}(3)^{\perp} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

$\mathfrak{su}(3)^{\perp}$ being the orthogonal complement of $\mathfrak{su}(3)$ in $\mathfrak{so}(6)$ and

$$\begin{aligned}\mathcal{W}_1 &= \mathcal{W}_1^+ \oplus \mathcal{W}_1^-, & \mathcal{W}_1^{\pm} &\cong \mathbb{R}, \\ \mathcal{W}_2 &= \mathcal{W}_2^+ \oplus \mathcal{W}_2^-, & \mathcal{W}_2^{\pm} &\cong \mathfrak{su}(3), \\ \mathcal{W}_3 &\cong \llbracket S^{2,0} \rrbracket, & \mathcal{W}_4 &\cong \mathcal{W}_5 \cong T^*,\end{aligned}$$

where $\llbracket S^{2,0} \rrbracket$ denotes the real representation associated with the space $S^{2,0}$ of complex symmetric tensors of type $(2, 0)$.

The components of the intrinsic torsion of an $SU(3)$ -structure can be expressed by (see e.g. [5, 1])

$$(13) \quad \begin{cases} d\omega &= \nu_0 \psi_+ + \alpha_0 \psi_- + \nu_1 \wedge \omega + \nu_3, \\ d\psi_+ &= \frac{2}{3}\alpha_0 \omega^2 + \pi_1 \wedge \psi_+ - \pi_2 \wedge \omega, \\ d\psi_- &= -\frac{2}{3}\nu_0 \omega^2 + J\pi_1 \wedge \psi_+ - \sigma_2 \wedge \omega, \end{cases}$$

where $\alpha_0 \in \mathcal{W}_1^+$, $\pi_1 \in \mathcal{W}_5$, $\pi_2 \in \mathcal{W}_2^+$, $\nu_0 \in \mathcal{W}_1^-$, $\sigma_2 \in \mathcal{W}_2^-$, $\nu_1 \in \mathcal{W}_4$, $\nu_3 \in \mathcal{W}_3$.

By definition, an $SU(3)$ -structure is called *integrable* if the intrinsic torsion vanishes. In this case ω and ψ are both closed. Therefore, the intrinsic torsion measures the failure of the holonomy group of the Levi-Civita connection of g to reduce to $SU(3)$.

If (ω, ψ) is in the class \mathcal{W}_2^+ , then by using (13) and taking into account the conditions $d\omega = d\psi_- = 0$, we get that the components $\nu_0, \alpha_0, \sigma_2, \nu_3, \nu_1, \pi_1$ vanish and hence

$$d\psi_+ = -\pi_2 \wedge \omega,$$

with π_2 belonging to the space

$$(14) \quad \begin{aligned} \mathcal{W}_2^+ &\cong \{ \gamma \in \Lambda^2 \mid \gamma \wedge \psi_+ = 0, \quad *J\gamma = -\gamma \wedge \omega \} \\ &= \{ \gamma \in \Lambda^2 \mid J\gamma = \gamma, \quad \gamma \wedge \omega^2 = 0 \}. \end{aligned}$$

By [1] the scalar curvature $\text{scal}(g)$ of the metric g is given by:

$$\text{scal}(g) = -\frac{1}{2}|\pi_2|^2.$$

Let α be a closed 1-form on S^1 . Consider on the product $N \times S^1$, the generalized G_2 -structure defined by the structure form of even type ρ given by (12) with companion

$$\hat{\rho} = \alpha - \psi_- - \frac{1}{2}\omega^2 \wedge \alpha.$$

We have the following

Theorem 5.1. *Let (N, ω, ψ) be a 6-dimensional manifold endowed with an $SU(3)$ -structure. The structure form ρ , given by (12), defines a strongly integrable generalized G_2 -structure on $N \times S^1$ with respect to a 3-form H (non necessarily closed), i.e. ρ satisfies the conditions*

$$(15) \quad d_H \rho = d_H \hat{\rho} = 0$$

if and only if N is in the class \mathcal{W}_2^+ and $H = \pi_2 \wedge \alpha$.

Proof. By (15) we get

$$\begin{cases} d\omega + d(\psi_+ \wedge \alpha) - \frac{1}{6}d(\omega^3) + H \wedge \omega + H \wedge \psi^+ \wedge \alpha = 0, \\ d\hat{\rho} + H \wedge \hat{\rho} = -d\psi_- - \frac{1}{2}d(\omega^2 \wedge \alpha) + H \wedge \alpha - H \wedge \psi_- = 0. \end{cases}$$

This is equivalent to say:

$$(16) \quad \begin{cases} d\omega = 0, \\ d(\psi_+ \wedge \alpha) = -H \wedge \omega, \\ H \wedge \psi_+ \wedge \alpha = 0 \\ d\psi_- = H \wedge \alpha, \\ H \wedge \psi_- = 0. \end{cases}$$

Hence, in particular

$$d\psi_- = 0, \quad H \wedge \alpha = 0.$$

It follows that $H = S \wedge f\alpha$, with S a 2-form on N and f a function on S^1 . Since $d\omega = 0$, we obtain

$$d\psi^+ \wedge \alpha = -S \wedge \omega \wedge f\alpha,$$

we have that f has to be a constant k and

$$d\psi_+ = -kS \wedge \omega,$$

with $kS = \pi_2$. Since π_2 is a $(1,1)$ -form, then $\pi_2 \wedge \psi_{\pm} = 0$. Therefore, equations (16) are satisfied if and only if N belongs to the class \mathcal{W}_2^+ . \square

Note that H is closed if and only if $d\pi_2 = 0$.

Homogeneous examples of 6-dimensional manifolds with a $SU(3)$ -structure in the class \mathcal{W}_2^+ are given in [8]. There it was proved that the 6-dimensional nilmanifolds $\Gamma \backslash G$ which carry an invariant $SU(3)$ -structures in the class \mathcal{W}_2^+ are the torus, the \mathbb{T}^2 -bundle over \mathbb{T}^4 and the \mathbb{T}^3 -bundle over \mathbb{T}^3 associated with the following nilpotent Lie algebras

$$\begin{aligned} (0, 0, 0, 0, 0, 0), \\ (0, 0, 0, 0, 12, 13), \\ (0, 0, 0, 12, 13, 23), \end{aligned}$$

where the notation $(0, 0, 0, 0, 12, 13)$ means that the dual \mathfrak{g}^* of the Lie algebra \mathfrak{g} has a basis (e^1, \dots, e^6) such that $de^i = 0, i = 1, \dots, 4$, $de^5 = e^1 \wedge e^2$ and $de^6 = e^1 \wedge e^3$.

In [17] the \mathbb{T}^2 -bundle over \mathbb{T}^4 has been considered and it has been proved that it admits a $SU(3)$ -structure in the class \mathcal{W}_2^+ .

By [8] the \mathbb{T}^3 -bundle over \mathbb{T}^3 admits a family of $SU(3)$ -structures in the class \mathcal{W}_2^+ given by

$$\begin{aligned} \omega &= e^{16} + \mu e^{25} + (\mu - 1)e^{34}, \\ \psi_+ &= (1 - \mu)e^{124} + \mu e^{135} - \mu(\mu - 1)e^{456} - e^{236}, \\ \psi_- &= -\mu(1 - \mu)e^{145} + (\mu - 1)e^{246} + \mu e^{356} + e^{123}, \end{aligned}$$

where μ is a real number different from 0 and 1. Such a family of $SU(3)$ -structures belongs to the class \mathcal{W}_2^+ with

$$\pi_2 = \mu^2 e^{25} - (\mu - 1)^2 e^{36} - e^{14},$$

and $d\pi_2 \neq 0$.

Manifolds in the class \mathcal{W}_2^+ can be also obtained as hypersurfaces of 7-dimensional manifolds with a G_2 -structure. The \mathbb{T}^2 -bundle over \mathbb{T}^4 can be also be viewed as a hypersurface of a 7-dimensional manifold with a calibrated G_2 -structure, i.e. such that the associated stable 3-form is closed. Indeed, if (M, φ) is a 7-dimensional manifold with a calibrated G_2 -structure, then any hypersurface $\iota : N \hookrightarrow M$ with unit normal vector ν such that the Lie derivative $L_\nu \varphi = 0$ admits an $SU(3)$ -structure (ω, ψ) in the class \mathcal{W}_2^+ defined by

$$\begin{aligned} \omega &= \nu \lrcorner \varphi, \\ \psi_+ &= \nu \lrcorner * \varphi, \\ \psi_- &= \iota^* \varphi. \end{aligned}$$

For general theory on an oriented hypersurface of a 7-dimensional manifold endowed with a G_2 -structure see [2].

If we consider the 7-dimensional nilmanifold associated with the Lie algebra (see [9])

$$(0, 0, 0, -13, -23, 0, 0)$$

and the hypersurface which is a maximal integral submanifold of the involutive distribution defined by the 1-form e^6 , then one gets the $SU(3)$ -structure considered above.

Another example of hypersurface (non nilmanifold) can be obtained by the 7-dimensional compact manifold $M = X \times S^1$, where X is the compact solvmanifold considered by Nakamura (see [15]), associated with the solvable Lie algebra

$$(0, 12 - 45, -13 + 46, 0, 15 - 24, -16 + 34, 0)$$

and endowed with the G_2 -structure

$$\varphi = e^{147} + e^{357} - e^{267} + e^{136} + e^{125} + e^{234} - e^{456}.$$

The compact hypersurface, maximal integral submanifold of the involutive distribution defined by the 1-form e^7 , has an $SU(3)$ -structure in the class \mathcal{W}_2^+ .

We will show that, if the $SU(3)$ -structure is not integrable, then the 2-form π_2 cannot be closed. Indeed,

Proposition 5.2. *Let N be a 6-dimensional manifold endowed with an $SU(3)$ -structure (ω, ψ) in the class \mathcal{W}_2^+ . If π_2 is closed, then the $SU(3)$ -structure is integrable. In particular, the associated Riemannian metric g is Ricci flat.*

Proof. As already remarked, (ω, ψ) is in the class \mathcal{W}_2^+ if and only if

$$(17) \quad d\psi_+ = -\pi_2 \wedge \omega, \quad d\psi_- = d\omega = 0,$$

with π_2 satisfying the following relations

$$\begin{aligned} \pi_2 \wedge \psi_- &= 0, & *J\pi_2 &= -\pi_2 \wedge \omega \\ J\pi_2 &= \pi_2, & \pi_2 \wedge \omega^2 &= 0. \end{aligned}$$

By our assumption that π_2 is closed, (17) and the above definition of \mathcal{W}_2^+ (see (14)) we have

$$0 = d(\pi_2 \wedge \psi_+) = \pi_2 \wedge d\psi_+ = \pi_2 \wedge d\psi_+ = |\pi_2|^2 * 1.$$

Then $\pi_2 = 0$ and we get the result. \square

In particular, as a consequence we have that if (N, ω, ψ) is 6-dimensional manifold endowed with a (not integrable) $SU(3)$ -structure in the class \mathcal{W}_2^+ , the 3-form $H = \pi_2 \wedge \alpha$ on $N \times S^1$ cannot be closed.

Remark 5.3. It has to be noted that, in view of Proposition 5.2, for $SU(3)$ -manifolds in the class \mathcal{W}_2^+ , the two conditions

$$d\pi_2 = 0 \quad \text{and} \quad d\psi_+ = 0$$

are equivalent.

Furthermore, under the conditions of Proposition 5.2, the holonomy group of the metric on the manifold N can be properly contained in $SU(3)$. Indeed, for example, if one takes the 6-manifold $N = M^4 \times \mathbb{T}^2$, where $(M^4, \omega_1, \omega_2, \omega_3)$ is an hyper-Kähler manifold and \mathbb{T}^2 is a 2-dimensional torus, then an $SU(3)$ -structure is defined by

$$\begin{aligned} \omega &= \omega_1 + e^5 \wedge e^6, \\ \psi_+ &= \omega_2 \wedge e^5 - \omega_3 \wedge e^6, \\ \psi_- &= \omega_2 \wedge e^6 + \omega_3 \wedge e^5, \end{aligned}$$

where $\{e^5, e^6\}$ is an orthonormal coframe on \mathbb{T}^2 . Since

$$d\omega_i = 0, \quad i = 1, 2, 3, \quad de^5 = de^6 = 0,$$

we have

$$d\omega = 0, \quad d\psi_{\pm} = 0.$$

Therefore, the manifold N endowed with the $SU(3)$ -structure defined by (ω, ψ) belongs to the class \mathcal{W}_2^+ and the holonomy of the associated Riemannian metric is strictly contained in $SU(3)$, since the metric is a product.

Remark 5.4. Consider on $N \times \mathbb{R}$ the generalized G_2 -structure defined by the structure form ρ given by (12) and let H be a closed non-zero 3-form. If we drop the condition $d_H \hat{\rho} = 0$, then the $SU(3)$ -structure (ω, ψ) on N has to be in the class $\mathcal{W}_2^+ \oplus \mathcal{W}_2^- \oplus \mathcal{W}_5$ with

$$d\psi_+ = \pi_1 \wedge \psi_+ - \pi_2 \wedge \omega = -S \wedge \omega, \quad dS = 0.$$

Indeed, ρ is d_H -closed if and only if

$$\begin{cases} d\omega = 0, \\ d\psi_+ \wedge \alpha = -H \wedge \omega, \\ H \wedge \psi_+ \wedge \alpha = 0. \end{cases}$$

Setting

$$H = \tilde{H} + S \wedge \alpha,$$

with \tilde{H} and S a 3-form and a 2-form respectively on N , then one gets the equivalent conditions:

$$\begin{cases} d\omega = 0, \\ d\psi_+ = -S \wedge \omega, \\ \tilde{H} \wedge \psi_+ = \tilde{H} \wedge \omega = 0, \\ dS = d\tilde{H} = 0. \end{cases}$$

In terms of the components of the intrinsic torsion one has that $\nu_0, \alpha_0, \nu_1, \nu_3$ vanish and

$$d\psi_+ = -S \wedge \omega.$$

In contrast with the case of $SU(3)$ -manifolds in the class \mathcal{W}_2^+ (see Proposition 5.2), 6-dimensional compact examples of this type may exist, as showed by the following

Example 5.5. Consider the 6-dimensional nilpotent Lie algebra \mathfrak{l} with structure equations

$$(0, 0, 0, 0, 0, 25)$$

and the $SU(3)$ -structure given by

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi &= (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6). \end{aligned}$$

Let H be the closed 3-form

$$\begin{aligned} H &= -e^{457} + a_1(e^{124} - e^{456}) + a_2(e^{125} - e^{345}) - a_3(e^{134} - e^{156}) + a_4e^{135} + \\ &\quad a_5(e^{145} - e^{235}) + a_6(e^{145} + e^{246}) + a_7(e^{234} - e^{256}) + a_8e^{245}, \end{aligned}$$

with $a_i \in \mathbb{R}$, $i = 1, \dots, 8$. Then (ω, ψ) induces a structure form ρ on a compact quotient of $L \times \mathbb{R}$, where L is the simply connected nilpotent Lie group with Lie algebra \mathfrak{l} , by a uniform discrete subgroup. A straightforward computation shows that $d_H \rho = 0$.

REFERENCES

- [1] Bedulli L., Vezzoni L.: The Ricci tensor of J . *Geom. Phys.* **57** (2007), no. 4, 1125–1146.
- [2] Cabrera F. M.: $SU(3)$ -structures on hypersurfaces of manifolds with G_2 -structures, *Monatsh. Math.* **148** (2006), pp. 29–50.
- [3] Cabrera F. M.: Special almost Hermitian geometry, *J. Geom. Phys.* **55** (2005), pp. 450–470.
- [4] Chiossi S., Fino A.: Conformally parallel G_2 structures on a class of solvmanifolds, *Math. Z.* **252** (2006), pp. 825–848 .
- [5] Chiossi S., Salamon S.: The intrinsic torsion of $SU(3)$ and G_2 structures, *Differential geometry*, Valencia, 2001, World Sci. Publishing, River Edge, NJ (2002), pp. 115–133.
- [6] Conti D., Salamon S.: Generalized Killing spinors in dimension 5, *Trans. AMS* **359** (2007), pp. 5319–5343.
- [7] Conti D., Tomassini A.: Special symplectic six manifolds, *Quart. J. Math.* **58** (2007), pp. 297–311.
- [8] de Bartolomeis P., Tomassini A.: On the Maslov Index of Lagrangian Submanifolds of Generalized Calabi-Yau Manifolds, *Internat. J. of Math.* **17** (2006), pp. 921–947.
- [9] Fernández M.: An example of compact calibrated manifold associated with the exceptional Lie group, *J. Differential Geom.* **26** (1987), pp. 367–370.
- [10] Gauntlett J. P., Martelli D., Pakis S., Waldram D.: G -structures and Wrapped NS5-Branes, *Comm. Math. Phys.* **247** (2004), pp. 421–445.
- [11] Harvey R., Lawson H. Blaine Jr.: Calibrated geometries, *Acta Math.* **148** (1982), pp. 47–157.
- [12] Hitchin N.J.: *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 70–89, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.
- [13] Hitchin N.J.: Generalized Calabi-Yau Manifolds, *Quart. J. Math.* **54** (2003), pp. 281–308.
- [14] Jeschek C., Witt F.: Generalised G_2 -structures and type IIB superstrings, *J. High Energy Phys.* (2005), 15 pp.
- [15] Nakamura I.: Complex parallelisable manifolds and their small deformations, *J. Differential Geom.* **10** (1975), pp. 85–112.
- [16] Tomassini A., Vezzoni L.: Contact Calabi-Yau manifolds and Special Legendrian submanifolds, e-print math.DG/0612232, to appear in *Osaka J. of Math.*
- [17] Witt F.: Generalised G_2 -manifolds, *Comm. Math. Phys.* **265** (2006), pp. 275–303.
- [18] Witt F.: Closed forms and special metrics, DPhil thesis, University of Oxford, 2004, e-print math.DG/0502443.

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